

TWO PROBLEMS FOR ONE HYPERBOLIC EQUATION OF THE THIRD ORDER IN THREE-DIMENSIONAL SPACE

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Abstract

In the present article, a modified Cauchy problem (problem C) for the hyperbolic equation of the third order with the data on the equation's coefficients singularity plane is solved by Riemann method. The special class in which the solution of the problem C has more simple appearance is introduced and the area of values of the parameter p entering into the equation is considerably expanded. In the special class the mixed problem, which decision was been reduced to the two-dimensional Volterra's integral equations of the first order with uncurtailed operators, is considered. Authors found the unique solution of these equations at various values of the parameter p .

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1 Introductory notes

In the present paper, the following equation is considered:

$$L(U) = U_{xyz} - \frac{p}{x-y-z}U_{xz} + \frac{p}{x-y-z}U_{yz} - \lambda U_z = 0 \quad (1)$$

($p, \lambda = \text{const}$) in the region $\mathcal{H} = \{(x, y, z) : 0 < z < x - y, 0 < y < x < +\infty\}$ of 3-dimensional Euclid space. Using Riemann method, for the equation (1), the solution of modified Cauchy problem has been obtained for the case $0 < p < \frac{1}{2}$ (Problem C). Special representation has been introduced – special class W_p – of the solutions of the problem C for equation (1) with the purpose:

- 1) to simplify the solution form of the problem C , making it more convenient to solve new boundary value problems;
- 2) expand the solution of the problem C to the case of negative values of the parameter for which Riemann method is not efficient.

The authors have obtained such solution of the mixed boundary value problem for various values of the parameter p .

Note that the plane analogue of the equation (1) with corresponding problems formulation has been considered in the paper [1].

2 Problem C

In the region \mathcal{H} , find solution of the equation (1), continuous in $\overline{\mathcal{H}}$, satisfying the conditions:

$$U(x, y, x - y) = \tau(x, y), \quad 0 \leq y \leq x < +\infty; \quad (2)$$

$$\lim_{z \rightarrow x-y-0} \frac{\partial U}{\partial z} = \nu(x, y), \quad 0 < y < x < +\infty; \quad (3)$$

$$\lim_{z \rightarrow x-y-0} (x - y - z)^{2p} (U_{xz} - U_{yz}) = \mu(x, y), \quad 0 < y < x < +\infty \quad (0 < p < \frac{1}{2}). \quad (4)$$

To solve the problem C, we apply Riemann method. Riemann function for the equation (1) has been built in the paper [2]. Represent it in the form:

$$V(x, y, z; x_0, y_0, z_0) = \frac{(x - y - z)^{2p}}{(x - y_0 - z)^p (x_0 - y - z)^p} \times \\ \times \sum_{m=0}^{\infty} \frac{[\lambda(x_0 - x)(y_0 - y)]^m}{(1)_m m!} F\left(p, p + m, 1 + m; \frac{(x - x_0)(y_0 - y)}{(x - y_0 - z)(x_0 - y - z)}\right), \quad (5)$$

where

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

In the region \mathcal{H} , take arbitrary point $M_0(x_0, y_0, z_0)$ and, because the coefficients of the equation (1)

$$a = \frac{-p}{x - y - z}, \quad b = \frac{p}{x - y - z} \quad (6)$$

become infinity on the plane $z = x - y$, let us consider the region H_ε limited by the planes $x = x_0$, $y = y_0$, $z = z_0$, $z = x - y - \varepsilon$ ($\varepsilon > 0$).

Supposing that the solution of the problem C exists, integrate Green identity derived in paper [2],

$$VL(U) - UL^*(v) = \frac{1}{3} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} + \frac{\partial H}{\partial x} \right] \quad (7)$$

over the region H_ε . For the equation (1), we have

$$P = VU_{yz} + UV_{yz} - V_y U_z + 3U_z a V, \quad (8)$$

$$Q = V(U_{xz} - 3U b_z) + UV_{xz} - V_z U_x - 3b UV_z, \quad (9)$$

$$H = V(U_{xy} - 3a_x U + 3b U_y) - V_x U_y - 3a UV_x + V_{xy} U - 3\lambda U, \quad (10)$$

where V is Riemann function (5); $U(x, y, z)$ – solution of the equation (1); $L^*(v)$ – conjugated operator; coefficients a and b are defined by the formula (6).

Applying Gauss–Ostrogradski formula to the obtained integral identity, we obtain

$$\sum_{i=0}^3 \int \int_{D_i} (P \cos \alpha + Q \cos \beta + H \cos \gamma) dS = \sum_{k=1}^4 J_k = 0. \quad (11)$$

Here, D_i ($i = \overline{0, 3}$) – the side of the pyramid H_ε located respectively in the planes $x = x_0$, $y = y_0$, $z = z_0$, $z = x - y - \varepsilon$. Consider each term of the formula (11)

$$J_1 = \int_{D_0} \int P|_{x=x_0} dy dz = \int_{z_0}^{x_0-y_0-\varepsilon} dz \int_{y_0}^{x_0-z-\varepsilon} (VU_{yz} + UV_{yz} - V_y U_z + 3U_z aV)|_{x=x_0} dy dz = \sum_{k=1}^4 i_k. \quad (12)$$

In the first two terms, i_1 and i_2 of eq. (12) we integrate by parts taking into account the property of Riemann function

$$(V_y = aV)_{x=x_0} = 0, \quad (13)$$

hence obtain

$$J_1 = U(x_0, y_0, z_0) - U(x_0, y_0, x_0 - y_0 - \varepsilon) + \int_{z_0}^{x_0-z_0-\varepsilon} U_z(x_0, x_0 - z - \varepsilon, z) V dz + \\ + \int_{y_0}^{x_0-z_0-\varepsilon} U_z(x_0, y, x_0 - y - \varepsilon) V_y dy - \int_{y_0}^{x_0-z_0-\varepsilon} U(x_0, y, z_0) V_y dy. \quad (14)$$

Integrating by parts, and also recalling the relations obtained by direct computation

$$(V_{xz} - b_z V - bV_z)_{y=y_0} = 0, \quad (15)$$

$$aV_x + a_x V + (bV - V_x)_y + \lambda V = 0, \quad (16)$$

we obtain the following results:

$$J_2 = - \int_{z_0}^{x_0-y_0-\varepsilon} dz \int_{z+y_0+\varepsilon}^{x_0} Q(x, y_0, z) dx = - \int_{z_0+y+\varepsilon}^{x_0} U_x(x, y_0, x - y_0 - \varepsilon) V dx + \\ + \int_{z_0+y_0+\varepsilon}^{x_0} U_x(x, y_0, z_0) V dx - 2 \int_{z_0}^{x_0-y_0-\varepsilon} U(z + y_0 + \varepsilon, y_0, z) V_z dz, \quad (17)$$

$$J_3 = - \int_{y_0}^{x_0-z_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} H(x, y, z_0) dx = - \int_{y_0}^{x_0-z_0-\varepsilon} U_y(x_0, y, z_0) V dy + \\ + \int_{y_0}^{x_0-z_0-\varepsilon} U_y(z_0 + y + \varepsilon, y, z_0) V dy - \int_{y_0+z_0+\varepsilon}^{x_0} U(x, x - z_0 - \varepsilon, z_0) V_x dx + \\ + \int_{y_0+z_0+\varepsilon}^{x_0} U(x, y_0, z_0) V_x dx - 3 \int_{y_0+z_0+\varepsilon}^{x_0} U(x, x - z_0 - \varepsilon, z_0) (bV - V_x) dx. \quad (18)$$

Into the integral $J_4 = \int_{y_0}^{x_0-z_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} (-P + Q + H)|_{z=x-y-\varepsilon} dx$, we replace P , Q , H with their values respectively from equations (8), (9), (10) and perform a series

of transformations aimed on removing the terms from the double integral containing $U(x, y, x - y - \varepsilon)$. This is achieved by integrating by parts recalling eq. (16). As a result, we obtain

$$\begin{aligned}
J_4 = & \int_{y_0}^{x_0-y_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} \left[\frac{3}{2}V(U_{xz} - U_{yz}) + 3(b-a)VU_z + \frac{3}{2}(V_y - V_x)U_z \right]_{z=x-y-\varepsilon} dx + \\
& + \frac{1}{2} \int_{y_0}^{x_0-z_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} V d_x(U_z) + \frac{1}{2} \int_{y_0}^{x_0-z_0-\varepsilon} \int_{z_0+y+\varepsilon}^{x_0} V d_y(U_z) dx + \\
& + \frac{1}{2} \int_{y_0}^{z_0+y+\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} (U_z V_y + V_x U_z) dx + \int_{y_0+z_0+\varepsilon}^{x_0} U(x, y_0; x - y_0 - \varepsilon) V_x dx \\
& - \int_{y_0+z_0+\varepsilon}^{x_0} U(x, x - z_0 - \varepsilon; z_0) V_x dx - \int_{y_0}^{x_0-z_0-\varepsilon} U(x_0, y; x_0 - y - \varepsilon) V_y dy \\
& + \int_{y_0}^{x_0-z_0-\varepsilon} U(z_0 + y + \varepsilon, y; z_0) V_y dy + 3 \int_{y_0+z_0+\varepsilon}^{x_0} U(x, x - z_0 - \varepsilon, z_0) b V dx - \\
& - 3 \int_{y_0+z_0+\varepsilon}^{x_0} U(x, y_0, x - y_0 - \varepsilon) b V dx + \int_{y_0+z_0+\varepsilon}^{x_0} U_x(x, x - z_0 - \varepsilon, z_0) V dx - \\
& - \int_{y_0+z_0+\varepsilon}^{x_0} U_x(x, y_0, x - y_0 - \varepsilon) V dx.
\end{aligned} \tag{19}$$

The first term of eq. (19) will be left intact, as for the 2nd and 3rd, they will be integrated by parts. As a result, all double integrals in eq. (19), except the first will mutually cancel. Substitution of the obtained result and the data of the equations (14), (17), (18) into the identity (11), cancels the single integrals by integrating by parts. The identity (11) takes the form of

$$\begin{aligned}
& 3U(x_0, y_0, z_0) - 3U(x_0, y_0, x_0 - y_0 - \varepsilon) + \frac{3}{2} \int_{z_0}^{x_0-y_0-\varepsilon} U_z(x_0, x_0 - z - \varepsilon, z) \frac{\varepsilon^p}{(x_0 - y_0 - z)^p} dz + \\
& + \frac{3}{2} \int_{z_0}^{x_0-y_0-\varepsilon} U_z(z_0 + y_0 + \varepsilon, y_0, z) \frac{\varepsilon^p}{(x_0 - y_0 - z)^p} dz + \\
& + \frac{3}{2} \int_{y_0}^{x_0-z_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} V(U_{xz} - U_{yz})|_{z=x-y-\varepsilon} dx + \\
& + 3 \int_{y_0}^{x_0-z_0-\varepsilon} dy \int_{z_0+y+\varepsilon}^{x_0} [(b-a)V + \frac{1}{2}(V_y - V_x)]U_z|_{z=x-y-\varepsilon} dx = 0.
\end{aligned} \tag{20}$$

Performing passage to the limit in the eq. (20) at $\varepsilon \rightarrow 0$ and recalling the conditions

(2)–(4) we obtain

$$\begin{aligned}
U(x_0, y_0, z_0) = & \tau(x_0, y_0) - \frac{\Gamma(1-2p)}{\Gamma^2(1-p)} \int_{y_0}^{x_0-z_0} ds \int_{z_0+s}^{x_0} \mu(t, s) \times \\
& \times (x_0 - t)^{-p} (s - y_0)^{-p} {}_0F_1(1-p; \lambda(x_0 - t)(y_0 - s)) dt - \\
& - \frac{\Gamma(2p)}{\Gamma^2(p)} \int_{y_0}^{x_0-z_0} ds \int_{z_0+s}^{x_0} \nu(t, s) (x_0 - t)^{p-1} (s - y_0)^{p-1} \times \\
& \times (x_0 - y_0 - t + s)^{1-2p} {}_0F_1(p; \lambda(x_0 - t)(y_0 - s)) dt, \\
{}_0F_1(\alpha, z) = & \sum_{n=0}^{\infty} \frac{z^n}{(\alpha)_n n!}.
\end{aligned} \tag{21}$$

Designate $D = \{(x, y) / 0 < y < x < +\infty\}$. Direct checking makes us certain that if

$$\tau_{xy} \in C(\overline{D}); \quad \nu \in C^{(2)}(\overline{D}); \quad \mu \in C^{(2)}(\overline{D}), \tag{22}$$

then the function (21) satisfies the equation (1) and conditions (2)–(4).

Theorem 1: At fulfillment of the conditions (22) the problem C for the equation (1) has unique solution represented by the formula (21).

3 Introduction of the special representation of the solution of the problem C

Let us introduce the special representation of the solution W_p of the problem C similar to what has been made by I.L. Karol [3] for the Euler–Darboux equation on a plane.

For the purpose of convenience of our further reasoning, let us convert the formula of the solution of the problem C (21): rename the variables $x_0 = x$, $y_0 = y$, $z_0 = z$, $s = s_1$; let us alternate the integration sequence in both integrals and do the replacement $s_1 = t - s$, after that alternate the integrating sequence again. The solution of the problem C will come to the form

$$\begin{aligned}
U(x, y, z) = & \tau(x, y) - \frac{1}{2} \frac{\Gamma(1-2p)}{\Gamma^2(1-p)} \int_z^{x-y} ds \int_{y+s}^x \mu(t, t-s) \times \\
& \times (x-t)^{-p} (t-y-s)^{-p} {}_0F_1(1-p; -\lambda(x-t)(t-y-s)) dt - \\
& - \frac{\Gamma(2p)}{\Gamma^2(p)} \int_z^{x-y} (x-y-s)^{1-2p} ds \int_{y+s}^x \nu(t, t-s) (x-t)^{p-1} (t-y-s)^{p-1} \times \\
& \times {}_0F_1(p; -\lambda(x-t)(t-y-s)) dt.
\end{aligned} \tag{23}$$

To simplify the formula (23), let us demand from the given function ν of integral [1] representation:

$$\nu(t, t-s) = \int_0^{t-s} T(s, \xi) (t-\xi-s)^{-2p} {}_0F_1(1-p, \lambda(t-s-\xi)^2) d\xi. \tag{24}$$

Definition: It is said that the solution (23) of the problem C for the equation (1) has special representation W_p if the function ν in it is defined by the equation (24) where $T(x, y)$ is a new function, $T(x, y) \in C^{(2)}(\overline{D})$.

To find the form of the solution W_p , substitute the function (24) into the expression (23) and transform the third term. Let us alternate the integrating sequence in it over t and ξ

$$\int_{y+s}^x dt \int_0^{t-s} d\xi = \int_0^y d\xi \int_{y+s}^x dt + \int_y^{x-s} d\xi \int_{\xi+s}^x dt$$

and consider the internal integrals

$$J_1 = \int_{y+s}^x (x-t)^{p-1} (t-y-s)^{p-1} (t-\xi-s)^{-2p} \times \\ \times {}_0F_1(p, -\lambda(x-t)(t-y-s)) {}_0F_1(1-p; \lambda(t-\xi-s)^2) dt.$$

Let us represent the functions ${}_0F_1(\alpha, z) = \sum_{n=0}^{\infty} \frac{z^n}{(\alpha)_n n!}$ with the series and apply the rule on multiplication of the series, alternate the sequence of summation and integration

$$J_1 = \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-\lambda)^k (-1)^m}{m!(k-m)!(1-p)_m (p)_{k-m}} \int_{y+s}^x (x-t)^{p-1+k-m} \times \\ \times (t-s-y)^{p-1+k-m} (t-\xi-s)^{-2p+2m} dt.$$

The integral will be named as i ; using the transformation $t = x - (x-y-s)\mu$, express it via hypergeometric Gauss function

$$i = \frac{\Gamma^2(p)}{\Gamma(2p)} \frac{[(p)_{k-m}]^2}{(2p)_{2k-2m}} (x-y-s)^{2p-1+2k-2m} \times \\ \times (x-\xi-s)^{-2p+2m} F\left(2p-2m, p+k-m; 2p+2k-2m; \frac{x-y-s}{x-\xi-s}\right).$$

Name $\sigma = \frac{x-y-s}{x-\xi-s}$ and apply autotransform formula [4]

$$F(\alpha, \beta, \gamma, \sigma) = (1-\sigma)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; \sigma).$$

As a result, we have

$$J_1 = \frac{\Gamma^2(p)}{\Gamma(2p)} (x-y-s)^{2p-1} (x-\xi-s)^{-2p} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(1-p)_k k!} \times \\ \times (x-y-s)^{2k} (1-\sigma)^{k-p} \sum_{m=0}^k \frac{(-1)^m k! (p)_{k-m} (1-p)_k}{m! (1-p)_m (k-m)!} \times \\ \times \frac{\sigma^{-2m} (1-\sigma)^m}{(2p)_{2k-2m}} F(2k; p+k-m; 2p+2k-2m; \sigma).$$

Using mathematical induction, we prove that the internal finite sum does not depend on p and equals $\sum_{m=0}^k = (-1)^k \sigma^{-2k}$, then

$$J_1 = \frac{\Gamma^2(p)}{\Gamma(2p)} (x-y-s)^{2p-1} (x-\xi-s)^{-p} (y-\xi)^{-p} \times \\ \times {}_0F_1(1-p, \lambda(y-\xi)(x-\xi-s)). \quad (25)$$

Analogous reasoning makes it possible to obtain

$$J_2 = \int_{\xi+s}^x (x-t)^{p-1} (t-y-s)^{p-1} (t-\xi-s)^{-2p} {}_0F_1(p, -\lambda(x-t)(t-y-s)) \times \\ \times {}_0F_1(1-p, \lambda(t-\xi-s)^2) dt = \frac{\Gamma(1-2p)\Gamma(p)}{\Gamma(1-p)} (x-s-y)^{2p-1} (\xi-s)^{-p} \times \\ \times (x-\xi-s)^{-p} {}_0F_1(1-p, -\lambda(x-\xi-s)(\xi-s)). \quad (26)$$

Recalling (25), (26) into the third term of eq. (23) and substituting $t-s = \xi$ in the 2nd term containing $\mu(t, t-s)$ follows

$$U(x, y, z) = \\ = \tau(x, y) - \int_z^{x-y} ds \int_0^y T(s, \xi) (y-\xi)^{-p} (x-\xi-s)^{-p} {}_0F_1(1-p, \lambda(y-\xi)(x-\xi-s)) d\xi - \\ - \int_z^{x-y} ds \int_y^{x-s} N(s, \xi) (x-\xi-s)^{-p} (\xi-y)^{-p} {}_0F_1(1-p, \lambda(y-\xi)(x-\xi-s)) d\xi, \quad (27)$$

where

$$N(s, \xi) = \frac{1}{2 \cos \pi p} T(s, \xi) + \frac{\Gamma(1-2p)}{2\Gamma^2(1-p)} \mu(\xi+s, \xi). \quad (28)$$

We need to note once again that the solution of the problem C defined by the equation (21) is obtained for the positive values of parameter p while for the negative values the sense is lost. The formula (28) is valid also for $p = -q$. Really, with a direct check it is possible to show that the function

$$U(x, y, z) = \tau_1(x, y) - \int_z^{x-y} ds \int_0^y T_1(s, \xi) (y-\xi)^q (x-\xi-s)^q \times \\ \times {}_0F_1(1+q, \lambda(y-\xi)(x-\xi-s)) d\xi - \int_z^{x-y} ds \int_y^{x-s} N_1(s, \xi) (x-\xi-s)^q \times \\ \times (\xi-y)^q {}_0F_1(1+q, \lambda(y-\xi)(x-\xi-s)) d\xi, \quad (29)$$

where

$$N_1(s, \xi) = \frac{1}{2 \cos \pi q} T_1(s, \xi) + \frac{\Gamma(1+2q)}{2\Gamma^2(1+q)} \mu_1(\xi+s, \xi) \quad (30)$$

is the solution of the equation (1) at $p = -q$ $\left(0 < q < \frac{1}{2}\right)$, which satisfies the conditions

$$U(x, y, x-y) = \tau_1(x, y), \quad 0 \leq y \leq x < +\infty; \quad (31)$$

$$\lim_{z \rightarrow x-y-0} \frac{\partial U}{\partial z} = \nu_1(x, y), \quad 0 < y < x < +\infty; \quad (32)$$

$$\lim_{z \rightarrow x-y-0} (x-y-z)^{-2q} (U_{xz} - U_{yz}) = \mu_1(x, y), \quad 0 < y < x < +\infty, \quad (33)$$

if the representation

$$\nu_1(t, t-s) = \int_0^{t-s} T_1(s, \xi) (t-\xi-s)^{2q} {}_0F_1(1+q, \lambda(t-s-\xi)^2) d\xi, \quad (34)$$

takes place where

$$\tau_{1xy}'' \in C(\overline{D}), \quad \nu_1, \mu_1, T_1 \in C(\overline{D}).$$

As already stated, in the case of negative parameter Riemann method proves to be inefficient, and even at $\lambda = 0$, when there is a possibility of obtaining the solution of Cauchy problem from the general solution of the equation, it has complicated structure which in essence has lead to the necessity of introduction of special representations [3].

Using the formulae (27), (29), it is possible to obtain the solutions of new boundary value problems: Cauchy–Goursat, Darboux, problems with integral conditions, and also problems with pasting together on a singularity plane of the equation coefficients or on characteristic plane.

4 Solution of the mixed problem in the special representation W_p

As an example, let us consider the solution of one mixed problem for the equation (1).

Problem $C - G$: (Cauchy–Goursat) in the region \mathcal{H} to find the solution of the equation (1) continuous in $\overline{\mathcal{H}}$, belonging to the class W_p , satisfying boundary conditions (2), (3) and the condition

$$U(x, 0, z) = \varphi(x, z), \quad 0 \leq z \leq x < +\infty \quad (35)$$

Let us suppose that at $0 < p < \frac{1}{2}$

$$\begin{aligned} \tau_{x,y} &\in C(\overline{D}), \quad \nu_{x,y} \in C^{(2)}(\overline{D}); \\ \tau(x, 0) &= 0, \quad \varphi''_{xz} \in C(\overline{D}), \quad \varphi(x, x) = \varphi'_z(z, z) = 0. \end{aligned} \quad (A)$$

Recall the equation (27) that is the solution of the problem C for eq. (1). This solution satisfies the conditions (2), (3). We shall find the unknown function $N(s, \xi)$ by assuming in the equation (27) $y = 0$ and subordinating it the condition (35)

$$-\int_z^x ds \int_0^{x-s} N(s, \xi) \xi^{-p} (x - \xi - s)^{-p} {}_0F_1(1 - p, -\lambda(x - \xi - s)\xi) d\xi = \varphi(x, z). \quad (36)$$

The problem $C - G$ has reduced to integral equation in $N(s, \xi)$. Supposing that the solution of the equation (36) exists, let us differentiate both sides of the identity (36)

by z , and then apply operator $K_{p,x,z}[u] = \int_z^{x+z} u(t, z) (x + z - t)^{p-1} dt$ to both sides of the obtained equation preliminarily replacing x with t in eq. (36).

As a result of calculations, obtain

$$\int_0^x N_1(z, \xi) {}_0F_1(1, -\lambda\xi(x - \xi)) d\xi = f(x, z), \quad (37)$$

where

$$N_1 = \xi^{-p} N, \quad f(x, z) = \frac{1}{\Gamma(p)\Gamma(1-p)} \int_z^{x+z} \varphi'_z(t, z) (x + z - t)^{p-1} dt. \quad (38)$$

The unique solution of the equation (37) has been obtained in the paper [5]

$$N_1(z, x) = f'_x(x, z) - x^{-1} \int_0^x f'_t(t, z) t \frac{\partial}{\partial t} {}_0F_1(1, \lambda x(x-t)) dt \quad (39)$$

on condition $f(0, z) = 0$.

Realling the expression of the function f from the formula (38) into the formula (37) and performing the calculations, we have

$$N(z, x) = \frac{x^p}{\Gamma(p)\Gamma(1-p)} \left[\int_z^{x+z} \varphi''_{zt}(t, z) (x+z-t)^{p-1} {}_0F_1(p, \lambda x(x-t+z)) dt - \right. \\ \left. - \frac{\lambda}{p} \int_z^{x+z} \varphi'_z(t, z) (x+z-t)^p {}_0F_1(1+p, \lambda x(x-t+z)) dt \right]. \quad (40)$$

Direct checking shows that at satisfying the conditions (A), the function (40) is the unique solution of eq. (37).

In case of negative parameter of the equation (1) $p = -q$, the solution of the problem $C - \mathcal{G}$ is reduced to integral equation in N_1 :

$$- \int_z^{x-y} ds \int_y^{x-s} N_1(s, \xi) (x-\xi-s)^q \xi^q {}_0F_1(1+q, -\lambda(\xi-s)(x-\xi-s)) d\xi = \varphi_1(x, z). \quad (41)$$

Supposing that a solution of the equation (41) exists, let us differentiate the identity (41) first with respect to z , then to x and apply operator $K_{q,x,z}[u]$ to both parts of the obtained equation. Then, reasoning in a similar way as in the case of eq. (36), it is possible to prove that the function

$$N_1(z, x) = \frac{x^{-q}}{\Gamma(1+q)\Gamma(1-q)} \left[\int_z^{x+z} \varphi'''_{1ztt}(t, z) (x+z-t)^{-q} {}_0F_1(1-q, \lambda x(x-t+z)) dt - \right. \\ \left. - \frac{\lambda}{1-q} \int_z^{x+z} \varphi''_{1zt}(t, z) (x+z-t)^{1-q} {}_0F_1(2-q, \lambda x(x-t+z)) dt \right] \quad (42)$$

is the unique solution of the eq. (41) at complimentary to (A) conditions superimposed on the given function φ_1 which have the form:

$$\varphi''_{1xzx} \in C(\overline{D}), \quad \varphi''_{1zx}(z, z) = 0. \quad (43)$$

Theorem 2: The problem $C - \mathcal{G}$ for eq. (1) has a unique solution defined by the equations (27), (40) at fulfilment of conditions (A) in case $0 < p < \frac{1}{2}$ and defined by the formulae (29), (42) at fulfilment of conditions (A) and (43) for $p = -q$ $\left(0 < q < \frac{1}{2}\right)$.

References

- [1] Dolgoplov V.M., Dolgoplov M.V., Rodionova I.N. Construction of Special Classes of Solutions for Some Differential Equations of Hyperbolic Type. Doklady Mathematics, 2009, V. 80, No. 3, P. 860-866.

- [2] Volkodavov V.F., Nikolayev N.Ya., Bystrova O.K., Zakharov V.N. Riemann functions for some differential equations in n -dimensional Euclidian space and their application. Samara University Press, 1995. P. 4–34.
- [3] Karol I.L. On one boundary value problem for the mixed elliptic-hyperbolic type differential equation. Proc. Acad. Sci. USSR. 1953. T.88. №2. C. 197.
- [4] Batemen G., Erdelyi A. Higher transcendent functions. V.I. Moscow: Nauka, 1973. P. 76.
- [5] Bushkov S.V., Rodionova I.N. Solution of the integral Volterra equations with the ${}_0F_1$ function in the kernels. Proc. Samara State Pedagogical university: Institute of Mathematics, Physics, and Informatics. Samara №1/2007. P. 19–25.